

Some q -analogues of (super)congruences of Beukers, Van Hamme and Rodriguez-Villegas

Victor J. W. Guo¹ and Jiang Zeng²

¹Department of Mathematics, Shanghai Key Laboratory of PMMP, East China Normal University,
500 Dongchuan Road, Shanghai 200241, People's Republic of China
jwguo@math.ecnu.edu.cn, [htap://math.ecnu.edu.cn/~jwguo](http://math.ecnu.edu.cn/~jwguo)

²Université de Lyon; Université Lyon 1; Institut Camille Jordan, UMR 5208 du CNRS;
43, boulevard du 11 novembre 1918, F-69622 Villeurbanne Cedex, France
zeng@math.univ-lyon1.fr, [htap://math.univ-lyon1.fr/~zeng](http://math.univ-lyon1.fr/~zeng)

Abstract. For any odd prime p we obtain q -analogues of Van Hamme's supercongruence:

$$\sum_{k=0}^{\frac{p-1}{2}} \binom{2k}{k}^3 \frac{1}{64^k} \equiv 0 \pmod{p^2} \quad \text{for } p \equiv 3 \pmod{4},$$

and Rodriguez-Villegas' Beukers-like supercongruences involving products of three binomial coefficients. For example, we prove that

$$\sum_{k=0}^{\frac{p-1}{2}} \left[\begin{matrix} 2k \\ k \end{matrix} \right]_{q^2}^3 \frac{q^{2k}}{(-q^2; q^2)_k^2 (-q; q)_{2k}^2} \equiv 0 \pmod{[p]^2} \quad \text{for } p \equiv 3 \pmod{4},$$

$$\sum_{k=0}^{\frac{p-1}{2}} \left[\begin{matrix} 2k \\ k \end{matrix} \right]_{q^3} \frac{(q; q^3)_k (q^2; q^3)_k q^{3k}}{(q^6; q^6)_k^2} \equiv 0 \pmod{[p]^2} \quad \text{for } p \equiv 2 \pmod{3},$$

where $[p] = 1 + q + \cdots + q^{p-1}$, $(a; q)_n = (1 - a)(1 - aq) \cdots (1 - aq^{n-1})$, and $\left[\begin{matrix} n \\ k \end{matrix} \right]_q$ denotes the q -binomial coefficient. Actually, our results give q -analogues of Z.-H. Sun's and Z.-W. Sun's generalizations of the above Beukers-like supercongruences. Our proof uses the theory of basic hypergeometric series including a new q -Clausen-type summation formula.

Keywords: congruences, supercongruences, basic hypergeometric series, q -binomial theorem, q -Chu-Vandermonde formula, the Lagrange interpolation, the Newton interpolation

2000 Mathematics Subject Classifications: Primary 11B65, Secondary 05A10, 05A30

1. Introduction

In 1985, Beukers [9] made the following conjecture: for any odd prime p ,

$$\sum_{k=0}^{\frac{p-1}{2}} (-1)^k \binom{\frac{p-1}{2}}{k} \binom{2k}{k}^2 \frac{1}{16^k} \equiv \begin{cases} 4x^2 - 2p \pmod{p^2}, & \text{if } p = x^2 + y^2 \text{ with } x \text{ odd,} \\ 0 \pmod{p^2}, & \text{if } p \equiv 3 \pmod{4} \end{cases} \pmod{p^2}. \quad (1.1)$$

Beukers himself proved this congruence modulo p , which is equivalent to

$$\sum_{k=0}^{\frac{p-1}{2}} \binom{2k}{k}^3 \frac{1}{64^k} \equiv \begin{cases} 4x^2 \pmod{p}, & \text{if } p = x^2 + y^2 \text{ with } x \text{ odd,} \\ 0 \pmod{p}, & \text{if } p \equiv 3 \pmod{4} \end{cases} \pmod{p}. \quad (1.2)$$

Complete proofs of (1.1) have been given by Ishikawa [17], Ahlgren [1], and Mortenson [22]. It is interesting that Van Hamme [38] proved the following company congruence of (1.1).

$$\sum_{k=0}^{\frac{p-1}{2}} \binom{2k}{k}^3 \frac{1}{64^k} \equiv \begin{cases} 4x^2 - 2p \pmod{p^2}, & \text{if } p = x^2 + y^2 \text{ with } x \text{ odd,} \\ 0 \pmod{p^2}, & \text{if } p \equiv 3 \pmod{4} \end{cases} \pmod{p^2}. \quad (1.3)$$

Finite fields analogues of classical hypergeometric series [14] play important parts in [1, 2, 22].

Motivated by his joint work [10] with Candelas and de la Ossa on Calabi-Yau manifolds over finite fields, Rodriguez-Villegas [26] discovered numerically some Beukers-like supercongruences, such as, for any prime $p > 3$,

$$\sum_{k=0}^{p-1} \binom{3k}{k} \binom{2k}{k}^2 \frac{1}{108^k} \equiv 0 \pmod{p^2} \quad \text{if } p \equiv 2 \pmod{3}, \quad (1.4)$$

$$\sum_{k=0}^{p-1} \binom{4k}{k} \binom{2k}{k}^2 \frac{1}{256^k} \equiv 0 \pmod{p^2} \quad \text{if } p \equiv 5, 7 \pmod{8}, \quad (1.5)$$

$$\sum_{k=0}^{p-1} \binom{6k}{3k} \binom{3k}{k} \binom{2k}{k} \frac{1}{1728^k} \equiv 0 \pmod{p^2} \quad \text{if } p \equiv 3 \pmod{4}. \quad (1.6)$$

Note that Rodriguez-Villegas [26] made conjectures concerning the numbers in (1.4)–(1.6) for general p , which were finally proved by Mortenson [22] and Z.-W. Sun [32]. Recently, by using the properties of generalized Legendre polynomials, Z.-H. Sun [30], among other things, proved the following result.

Theorem 1.1. [30, Theorem 2.5] *Let p be an odd prime and let a be a p -adic integer with $\langle a \rangle_p \equiv 1 \pmod{2}$. Then*

$$\sum_{k=0}^{p-1} \binom{2k}{k} \binom{a}{k} \binom{-1-a}{k} \frac{1}{4^k} \equiv 0 \pmod{p^2}, \quad (1.7)$$

where $\langle a \rangle_p$ denotes the least nonnegative residue of a modulo p .

It is not difficult to see that, taking $a = -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, -\frac{1}{6}$ in (1.7), we obtain the $p \equiv 3 \pmod{4}$ case of (1.2), and (1.4)–(1.6). The first aim of this paper is to give a q -analogue of (1.3) and (1.7).

On the other hand, for any prime $p \geq 5$, Mortenson [20, 21] proved the following four supercongruences conjectured by Rodriguez-Villegas [26, (36)]:

$$\sum_{k=0}^{p-1} \binom{2k}{k}^2 \frac{1}{16^k} \equiv \left(\frac{-1}{p} \right) \pmod{p^2}, \quad (1.8)$$

$$\sum_{k=0}^{p-1} \binom{3k}{2k} \binom{2k}{k} \frac{1}{27^k} \equiv \left(\frac{-3}{p} \right) \pmod{p^2}, \quad (1.9)$$

$$\sum_{k=0}^{p-1} \binom{4k}{2k} \binom{2k}{k} \frac{1}{64^k} \equiv \left(\frac{-2}{p} \right) \pmod{p^2}, \quad (1.10)$$

$$\sum_{k=0}^{p-1} \binom{6k}{3k} \binom{3k}{k} \frac{1}{432^k} \equiv \left(\frac{-1}{p} \right) \pmod{p^2}, \quad (1.11)$$

where $\left(\frac{\cdot}{p} \right)$ denotes the Legendre symbol modulo p . Z.-H. Sun [30] gave an elementary proof of (1.8)–(1.11) by using generalized Legendre polynomials. See also [28, 34, 35] for several simple proofs of (1.8). Z.-W. Sun [33, (1.4)] obtained the following generalization of (1.8):

$$\sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k} \binom{2k+2s}{k+s}}{4^{2k+s}} \equiv \left(\frac{-1}{p} \right) \pmod{p^2}, \quad \text{for } 0 \leq s \leq \frac{p-1}{2}, \quad (1.12)$$

where $\left(\frac{\cdot}{p} \right)$ denotes the Legendre symbol modulo p . Note that McCarthy and Osburn [19] have studied some related interesting supercongruences.

Recall that the q -shifted factorials are defined by $(a; q)_0 = 1$ and

$$(a; q)_n = (1-a)(1-aq) \cdots (1-aq^{n-1}) \text{ for } n = 1, 2, \dots,$$

and the q -integer is defined as $[p] = \frac{1-q^p}{1-q}$. The q -binomial coefficients $\begin{bmatrix} n \\ k \end{bmatrix}$ are then defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{cases} \frac{(q^{n-k+1}; q)_k}{(q; q)_k}, & \text{if } 0 \leq k \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

In the last decade, several authors have studied q -analogues of congruences and supercongruences, see [5, 23, 25, 27, 36]. Throughout the paper we will often use the fact that for any prime p , the q -integer $[p]$ is always an irreducible polynomial in $\mathbb{Q}[q]$. Namely, $\mathbb{Q}[q]/[p]$ is a field. Therefore, rational functions $a(q)/b(q)$ are well defined modulo $[p]$ or $[p]^r$ ($r \geq 1$) on condition that $b(q)$ is relatively prime to $[p]$. In a previous paper [16], we

give some q -analogues of (1.8) and partial q -analogues of (1.9)–(1.11) such as

$$\sum_{k=0}^{p-1} \frac{(q; q^2)_k^2}{(q^2; q^2)_k^2} \equiv \left(\frac{-1}{p} \right) q^{\frac{1-p^2}{4}} \pmod{[p]^2}, \quad (1.13)$$

$$\sum_{k=0}^{p-1} \frac{(q; q^3)_k (q^2; q^3)_k}{(q^3; q^3)_k^2} \equiv \left(\frac{-3}{p} \right) q^{\frac{1-p^2}{3}} \pmod{[p]}, \quad (1.14)$$

$$\sum_{k=0}^{p-1} \frac{(q; q^4)_k (q^3; q^4)_k}{(q^4; q^4)_k^2} \equiv \left(\frac{-2}{p} \right) q^{\frac{3(1-p^2)}{8}} \pmod{[p]}, \quad (1.15)$$

$$\sum_{k=0}^{p-1} \frac{(q; q^6)_k (q^5; q^6)_k}{(q^6; q^6)_k^2} \equiv \left(\frac{-1}{p} \right) q^{\frac{5(1-p^2)}{12}} \pmod{[p]}. \quad (1.16)$$

The second aim of this paper is to give a q -analogue of (1.12) and a generalization of (1.13)–(1.16).

2. The main results

Motivated by Z.-W. Sun's generalization [32, Theorem 1.1(i)] of Van Hamme' congruence (1.3):

$$\sum_{k=0}^{\frac{p-1}{2}} \binom{2k}{k}^2 \binom{2k}{k+s} \frac{1}{64^k} \equiv 0 \pmod{p^2}, \quad (2.1)$$

where $0 \leq s \leq \frac{p-1}{2}$ and $s \equiv \frac{p+1}{2} \pmod{2}$, we give a unified q -analogue of (1.3) and (2.1) as follows.

Theorem 2.1. *Let p be an odd prime and $0 \leq s \leq \frac{p-1}{2}$. Then we have the following congruence modulo $[p]^2$:*

$$\begin{aligned} & \sum_{k=0}^{\frac{p-1}{2}} \left[\begin{matrix} 2k \\ k \end{matrix} \right]_{q^2}^2 \left[\begin{matrix} 2k \\ k+s \end{matrix} \right]_{q^2} \frac{q^{2k}}{(-q^2; q^2)_k^2 (-q; q)_{2k}^2} \\ & \equiv \begin{cases} (-1)^s q^{\frac{p-1}{2}-s^2} \left[\begin{matrix} \frac{p-1}{2} \\ \frac{p-2s-1}{4} \end{matrix} \right]_{q^4}^2 \frac{(q^2; q^2)_{\frac{p-2s-1}{2}} (q^2; q^2)_{\frac{p+2s-1}{2}}}{(q^4; q^4)_{\frac{p-1}{2}}^2}, & \text{if } s \equiv \frac{p-1}{2} \pmod{2}, \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \quad (2.2)$$

Letting $s = 0$ in (2.2), we immediately get the following neat q -analogue of (1.2).

Corollary 2.2. *Let p be an odd prime. Then we have the following congruence:*

$$\begin{aligned} & \sum_{k=0}^{\frac{p-1}{2}} \left[\begin{matrix} 2k \\ k \end{matrix} \right]_{q^2}^3 \frac{q^{2k}}{(-q^2; q^2)_k^2 (-q; q)_{2k}^2} \\ & \equiv \begin{cases} q^{\frac{p-1}{2}} \left[\begin{matrix} \frac{p-1}{2} \\ \frac{p-1}{4} \end{matrix} \right]_{q^4}^2 \frac{1}{(-q^2; q^2)_{\frac{p-1}{2}}^2}, & \text{if } p \equiv 1 \pmod{4}, \\ 0, & \text{otherwise,} \end{cases} \pmod{[p]^2}. \end{aligned} \quad (2.3)$$

Letting $q \rightarrow 1$ in (2.3), and using the congruence (see [29, Lemma 3.4])

$$\left(\frac{\frac{p-1}{2}}{\frac{p-1}{4}} \right)^2 \frac{1}{2^{p-1}} \equiv 4x^2 - 2p \pmod{p^2}, \quad (2.4)$$

where $p \equiv 1 \pmod{4}$ is a prime and $p = x^2 + y^2$ with $x \equiv 1 \pmod{4}$, we obtain Van Hamme's congruence (1.2). Note that (2.4) is easily proved by the Beukers-Chowla-Dwork-Evans congruence [11, 24]:

$$\left(\frac{\frac{p-1}{2}}{\frac{p-1}{4}} \right) \equiv \frac{2^{p-1} + 1}{2} \left(2x - \frac{p}{2x} \right) \pmod{p^2},$$

where the numbers p and x are the same as in (2.4).

Remark. Let p be an odd prime of the form $4k + 3$. Then by the antisymmetry of the k -th and $(\frac{p-1}{2} - k)$ -th terms modulo $[p]$, we can easily prove that

$$\sum_{k=0}^{\frac{p-1}{2}} \left[\begin{matrix} 2k \\ k \end{matrix} \right]_{q^2}^3 \frac{q^{k-2k^2}}{(-q^2; q^2)_k^2 (-q; q)_{2k}^2} \equiv 0 \pmod{[p]}.$$

Our second result is a unified q -analogue of Z.-H. Sun's congruence (1.7) and Z.-W. Sun's generalization of (1.4)–(1.6).

Theorem 2.3. *Let p be an odd prime and m, r two positive integers with $p \nmid m$. Let $s \leq \min\{\langle -\frac{r}{m} \rangle_p, \langle -\frac{m-r}{m} \rangle_p\}$ be a nonnegative integer. If $\langle -\frac{r}{m} \rangle_p \equiv s + 1 \pmod{2}$, then*

$$\sum_{k=s}^{\frac{p-1}{2}} \frac{(q^m; q^m)_{2k} (q^r; q^m)_k (q^{m-r}; q^m)_k q^{mk}}{(q^m; q^m)_{k-s} (q^m; q^m)_{k+s} (q^{2m}; q^{2m})_k^2} \equiv 0 \pmod{[p]^2}, \quad (2.5)$$

$$\sum_{k=s}^{p-1} \frac{(q^m; q^m)_{2k} (q^r; q^m)_k (q^{m-r}; q^m)_k q^{mk}}{(q^m; q^m)_{k-s} (q^m; q^m)_{k+s} (q^{2m}; q^{2m})_k^2} \equiv 0 \pmod{[p]^2}. \quad (2.6)$$

If $\langle -\frac{r}{m} \rangle_p \equiv s \pmod{2}$, then the following congruence holds modulo $[p]$:

$$\begin{aligned} & \sum_{k=s}^{\frac{p-1}{2}} \frac{(q^m; q^m)_{2k} (q^r; q^m)_k (q^{m-r}; q^m)_k q^{mk}}{(q^m; q^m)_{k-s} (q^m; q^m)_{k+s} (q^{2m}; q^{2m})_k^2} \\ & \equiv \frac{q^{(s+\frac{p-1}{2})m} (q^m; q^{2m})_{\langle -\frac{r}{m} \rangle_p - s} (q^m; q^{2m})_{\langle -\frac{m-r}{m} \rangle_p - s} (q^{-m\langle -\frac{r}{m} \rangle_p}; q^m)_s (q^{-m\langle -\frac{m-r}{m} \rangle_p}; q^m)_s}{(q^{2m}; q^{2m})_{\langle -\frac{r}{m} \rangle_p + s} (q^{2m}; q^{2m})_{\langle -\frac{m-r}{m} \rangle_p + s}}. \end{aligned} \quad (2.7)$$

Letting $s = 0$, $-\frac{r}{m} = a$ and $q \rightarrow 1$ in (2.6), we obtain (1.7). On the other hand, it is not difficult to see that (see [16]), for any prime $p \geq 5$,

$$(-1)^{\langle -\frac{1}{3} \rangle_p} = \left(\frac{-3}{p} \right), \quad (-1)^{\langle -\frac{1}{4} \rangle_p} = \left(\frac{-2}{p} \right), \quad (-1)^{\langle -\frac{1}{6} \rangle_p} = \left(\frac{-1}{p} \right). \quad (2.8)$$

Taking $r = 1$ and $m = 3, 4, 6$ in (2.5), we obtain

Corollary 2.4. *Let $p \geq 5$ be a prime and let s be a nonnegative integer. Then there hold the following congruences modulo $[p]^2$:*

$$\begin{aligned} & \sum_{k=s}^{\frac{p-1}{2}} \left[\begin{matrix} 2k \\ k+s \end{matrix} \right]_{q^3} \frac{(q; q^3)_k (q^2; q^3)_k q^{3k}}{(q^6; q^6)_k^2} \equiv 0, \text{ if } s \leq \frac{p-1}{3} \text{ and } s \equiv \frac{1 + (\frac{-3}{p})}{2} \pmod{2}, \\ & \sum_{k=s}^{\frac{p-1}{2}} \left[\begin{matrix} 2k \\ k+s \end{matrix} \right]_{q^4} \frac{(q; q^4)_k (q^3; q^4)_k q^{4k}}{(q^8; q^8)_k^2} \equiv 0, \text{ if } s \leq \frac{p-1}{4} \text{ and } s \equiv \frac{1 + (\frac{-2}{p})}{2} \pmod{2}, \\ & \sum_{k=s}^{\frac{p-1}{2}} \left[\begin{matrix} 2k \\ k+s \end{matrix} \right]_{q^6} \frac{(q; q^6)_k (q^5; q^6)_k q^{6k}}{(q^{12}; q^{12})_k^2} \equiv 0, \text{ if } s \leq \frac{p-1}{6} \text{ and } s \equiv \frac{1 + (\frac{-1}{p})}{2} \pmod{2}. \end{aligned}$$

The proof of Theorem 2.3 is based on the following highly non-trivial q -Clausen-type summation formula, which seems new and interesting in its own right.

Theorem 2.5. *Let n and s be nonnegative integers with $s \leq n$. Then*

$$\begin{aligned} & \left(\sum_{k=s}^n \frac{(q^{-2n}; q^2)_k (x; q)_k q^k}{(q; q)_{k-s} (q; q)_{k+s}} \right) \left(\sum_{k=s}^n \frac{(q^{-2n}; q^2)_k (q/x; q)_k q^k}{(q; q)_{k-s} (q; q)_{k+s}} \right) \\ & = \frac{(-1)^n (q^2; q^2)_n^2 q^{-n^2}}{(q^2; q^2)_{n-s} (q^2; q^2)_{n+s}} \sum_{k=s}^n \frac{(-1)^k (q^2; q^2)_{n+k} (x; q)_k (q/x; q)_k q^{k^2-2nk}}{(q^2; q^2)_{n-k} (q; q)_{k-s} (q; q)_{k+s} (q; q)_{2k}}. \end{aligned} \quad (2.9)$$

Recall that the *basic hypergeometric series* ${}_{r+1}\phi_r$ (see [13]) is defined by

$${}_{r+1}\phi_r \left[\begin{matrix} a_1, a_2, \dots, a_{r+1} \\ b_1, b_2, \dots, b_r \end{matrix} ; q, z \right] = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \dots, a_{r+1}; q)_n z^n}{(q, b_1, b_2, \dots, b_r; q)_n},$$

where $(a_1, \dots, a_m; q)_n = (a_1; q)_n \cdots (a_m; q)_n$. Theorem 2.5 is reminiscent to Jackson's q -analogue of Clausen's formula:

$${}_2\phi_1 \left[\begin{matrix} a, b \\ abq^{\frac{1}{2}} \end{matrix}; q, z \right] {}_2\phi_1 \left[\begin{matrix} a, b \\ abq^{\frac{1}{2}} \end{matrix}; q, zq^{\frac{1}{2}} \right] = {}_4\phi_3 \left[\begin{matrix} a, b, a^{\frac{1}{2}}b^{\frac{1}{2}}, -a^{\frac{1}{2}}b^{\frac{1}{2}} \\ ab, a^{\frac{1}{2}}b^{\frac{1}{2}}q^{1/4}, -a^{\frac{1}{2}}b^{\frac{1}{2}}q^{1/4} \end{matrix}; q^{\frac{1}{2}}, z \right].$$

We also have the following q -analogue of (1.12), which reduces to (1.13) when $s = 0$.

Theorem 2.6. *Let p be an odd prime and let $0 \leq s \leq \frac{p-1}{2}$. Then*

$$\sum_{k=0}^{\frac{p-1}{2}} \frac{(q; q^2)_k (q; q^2)_{k+s}}{(q^2; q^2)_k (q^2; q^2)_{k+s}} \equiv \left(\frac{-1}{p} \right) q^{\frac{1-p^2}{4}} \pmod{[p]^2}. \quad (2.10)$$

Finally, we have the following generalization of the previous congruences (1.14)–(1.16).

Theorem 2.7. *Let p be an odd prime and let m, r be positive integers with $p \nmid m$ and $r < m$. Then for any integer s with $0 \leq s \leq \langle -\frac{m-r}{m} \rangle_p$, there holds*

$$\sum_{k=0}^{p-s-1} \frac{(q^r; q^m)_k (q^{m-r}; q^m)_{k+s}}{(q^m; q^m)_k (q^m; q^m)_{k+s}} \equiv (-1)^{\langle -\frac{r}{m} \rangle_p} q^{\frac{-m \langle -\frac{r}{m} \rangle_p (\langle -\frac{r}{m} \rangle_p + 1)}{2}} \pmod{[p]}. \quad (2.11)$$

In particular, if $p \equiv \pm 1 \pmod{m}$, then

$$\sum_{k=0}^{p-s-1} \frac{(q^r; q^m)_k (q^{m-r}; q^m)_{k+s}}{(q^m; q^m)_k (q^m; q^m)_{k+s}} \equiv (-1)^{\langle -\frac{r}{m} \rangle_p} q^{\frac{r(m-r)(1-p^2)}{2m}} \pmod{[p]}. \quad (2.12)$$

By (2.8), letting $m = 3, 4, 6$ in (2.12) and noticing that

$$(-1)^{\langle -\frac{1}{3} \rangle_p} = (-1)^{\langle -\frac{2}{3} \rangle_p}, \quad (-1)^{\langle -\frac{1}{4} \rangle_p} = (-1)^{\langle -\frac{3}{4} \rangle_p}, \quad (-1)^{\langle -\frac{1}{6} \rangle_p} = (-1)^{\langle -\frac{5}{6} \rangle_p},$$

we obtain

Corollary 2.8. *Let $p > 3$ be a prime and $s \geq 0$. Then the following congruences hold modulo $[p]$:*

$$\sum_{k=0}^{p-s-1} \frac{(q^r; q^3)_k (q^{3-r}; q^3)_{k+s}}{(q^3; q^3)_k (q^3; q^3)_{k+s}} \equiv \left(\frac{-3}{p} \right) q^{\frac{1-p^2}{4}} \quad \text{for } r = 1, 2, \text{ and } s \leq \left\langle \frac{r-3}{3} \right\rangle_p, \quad (2.13)$$

$$\sum_{k=0}^{p-s-1} \frac{(q^r; q^4)_k (q^{4-r}; q^4)_{k+s}}{(q^4; q^4)_k (q^4; q^4)_{k+s}} \equiv \left(\frac{-2}{p} \right) q^{\frac{3(1-p^2)}{8}} \quad \text{for } r = 1, 3, \text{ and } s \leq \left\langle \frac{r-4}{4} \right\rangle_p, \quad (2.14)$$

$$\sum_{k=0}^{p-s-1} \frac{(q^r; q^6)_k (q^{6-r}; q^6)_{k+s}}{(q^6; q^6)_k (q^6; q^6)_{k+s}} \equiv \left(\frac{-1}{p} \right) q^{\frac{5(1-p^2)}{12}} \quad \text{for } r = 1, 5, \text{ and } s \leq \left\langle \frac{r-6}{6} \right\rangle_p. \quad (2.15)$$

3. Proof of Theorem 2.1

We first establish two lemmas.

Lemma 3.1. *Let p be an odd prime and $0 \leq k \leq \frac{p-1}{2}$. Then*

$$\left[\begin{matrix} \frac{p-1}{2} + k \\ 2k \end{matrix} \right]_{q^2} \equiv (-1)^k \left[\begin{matrix} 2k \\ k \end{matrix} \right]_{q^2} \frac{q^{kp-k^2}}{(-q; q)_{2k}^2} \pmod{[p]^2}. \quad (3.1)$$

Proof. Since

$$(1 - q^{p-2j+1})(1 - q^{p+2j-1}) + (1 - q^{2j-1})^2 q^{p-2j+1} = (1 - q^p)^2,$$

we have

$$(1 - q^{p-2j+1})(1 - q^{p+2j-1}) \equiv -(1 - q^{2j-1})^2 q^{p-2j+1} \pmod{[p]^2}.$$

It follows that

$$\begin{aligned} \left[\begin{matrix} \frac{p-1}{2} + k \\ 2k \end{matrix} \right]_{q^2} &= \frac{\prod_{j=1}^k (1 - q^{p-2j+1})(1 - q^{p+2j-1})}{(q^2; q^2)_{2k}} \\ &\equiv (-1)^k \frac{\prod_{j=1}^k (1 - q^{2j-1})^2 q^{p-2j+1}}{(q^2; q^2)_{2k}} \\ &= (-1)^k \left[\begin{matrix} 2k \\ k \end{matrix} \right]_q^2 \frac{q^{kp-k^2}}{(-q; q)_{2k}^2} \pmod{[p]^2}, \end{aligned}$$

as desired. \square

Lemma 3.2. *For nonnegative integers n and s such that $s \leq n$ we have*

$$\begin{aligned} &\sum_{k=0}^n \left[\begin{matrix} n+k \\ 2k \end{matrix} \right] \left[\begin{matrix} 2k \\ k \end{matrix} \right] \left[\begin{matrix} 2k \\ k+s \end{matrix} \right] \frac{(-1)^k q^{\binom{n-k}{2}}}{(-q; q)_k^2} \\ &= \begin{cases} (-1)^s q^{\frac{n^2-s^2}{2}} \left[\begin{matrix} n \\ \frac{n-s}{2} \end{matrix} \right]_{q^2}^2 \frac{(q; q)_{n-s} (q; q)_{n+s}}{(q^2; q^2)_n^2}, & \text{if } n \equiv s \pmod{2}, \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \quad (3.2)$$

Proof. We may rewrite the left-hand side of (3.2) as

$$\begin{aligned} &\sum_{k=s}^n \frac{(q^{-n}; q)_k (q^{n+1}; q)_k (q^{\frac{1}{2}}; q)_k (-q^{\frac{1}{2}}; q)_k q^{k+\binom{n}{2}}}{(q; q)_k (q; q)_{k-s} (q; q)_{k+s} (-q; q)_k} \\ &= \frac{(q^{-n}; q)_s (q^{n+1}; q)_s (q; q^2)_{2s} q^{s+\binom{n}{2}}}{(q^2; q^2)_s (q; q)_{2s}} {}_4\phi_3 \left[\begin{matrix} q^{s-n}, q^{n+s+1}, q^{s+\frac{1}{2}}, -q^{s+\frac{1}{2}} \\ q^{s+1}, -q^{s+1}, q^{2s+1} \end{matrix}; q, q \right]. \end{aligned}$$

The result then follows from Andrews' terminating q -analogue of Watson's formula [13, (II.17)]:

$${}_4\phi_3 \left[\begin{matrix} q^{-n}, a^2 q^{n+1}, b, -b \\ aq, -aq, b^2 \end{matrix}; q, q \right] = \begin{cases} 0, & \text{if } n \text{ is odd,} \\ \frac{b^n (q, a^2 q^2 / b^2; q^2)_{n/2}}{(a^2 q^2, b^2 q; q^2)_{n/2}}, & \text{if } n \text{ is even} \end{cases} \quad (3.3)$$

with the substitution of n, a and b by $n - s, q^s$ and $q^{s+\frac{1}{2}}$, respectively. \square

Proof of Theorem 2.1. By the congruence (3.1), the left-hand side of (2.2) is equal to

$$\begin{aligned} & \sum_{k=0}^{\frac{p-1}{2}} \begin{bmatrix} 2k \\ k \end{bmatrix}_{q^2}^2 \begin{bmatrix} 2k \\ k+s \end{bmatrix}_{q^2} \frac{q^{2k}}{(-q^2; q^2)_k^2 (-q; q)_{2k}^2} \\ & \equiv \sum_{k=0}^{\frac{p-1}{2}} \begin{bmatrix} \frac{p-1}{2} + k \\ 2k \end{bmatrix}_{q^2} \begin{bmatrix} 2k \\ k \end{bmatrix}_{q^2} \begin{bmatrix} 2k \\ k+s \end{bmatrix}_{q^2} \frac{(-1)^k}{(-q^2; q^2)_k^2} q^{k^2+2k-pk}. \end{aligned}$$

The proof then follows from (3.2) with $n = \frac{p-1}{2}$ and $q \rightarrow q^2$. \square

4. Proof of Theorem 2.5

We first establish four lemmas to make the proof easier. The following result can be derived from the Lagrange interpolation formula and the Newton interpolation formula (see [12]), we give a simple proof using the partial fraction decomposition technique as in [39].

Lemma 4.1. *Let n be a nonnegative integer. Then*

$$\sum_{k=0}^n (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(aq^n; q)_k q^{\binom{n-k+1}{2}}}{(a; q)_k (1 - xq^{-k})} = \frac{(ax; q)_n (q; q)_n}{(a; q)_n (xq^{-n}; q)_{n+1}}, \quad (4.1)$$

$$\sum_{j=0}^m (-1)^{m-j} \begin{bmatrix} m \\ j \end{bmatrix} q^{\binom{j}{2}} \frac{(q; q)_{m-j}}{(x; q)_{m-j+1}} = \frac{q^{\binom{m+1}{2}}}{1 - xq^m}. \quad (4.2)$$

Proof. For (4.1), by the partial fraction decomposition we have

$$\frac{(ax; q)_n (q; q)_n}{(a; q)_n (xq^{-n}; q)_{n+1}} = \sum_{k=0}^n \frac{a_k}{1 - xq^{-k}}$$

with

$$a_k = \lim_{x \rightarrow q^k} \frac{(1 - xq^{-k})(ax; q)_n (q; q)_n}{(a; q)_n (xq^{-n}; q)_{n+1}} = (-1)^{n-k} q^{\binom{n-k+1}{2}} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(aq^n; q)_k}{(a; q)_k}.$$

By the Gauss or q -binomial inversion (see, for example, [3, p. 77, Exercise 2.47]), the identity (4.2) is equivalent to

$$\frac{(q; q)_m}{(x; q)_{m+1}} = \sum_{k=0}^m \begin{bmatrix} m \\ k \end{bmatrix}_q (-1)^k q^{\binom{k+1}{2}} \frac{1}{1 - xq^k}, \quad (4.3)$$

which is routine by the partial fraction decomposition. \square

Lemma 4.2. *Let n be a positive integer. Then*

$$(x; q)_n + (a/x; q)_n = (x; q)_n (a/x; q)_n + \sum_{k=0}^{n-1} \frac{(x; q)_k (a/x; q)_k (1 - q^n)}{(q; q)_k (1 - q^{n-k})} \sum_{j=0}^k (-1)^j \begin{bmatrix} k \\ j \end{bmatrix} q^{\binom{j}{2}} (aq^{k+j}; q)_{n-k}, \quad (4.4)$$

$$(x; q)_n + (a/x; q)_n = (x; q)_n (a/x; q)_n + (a; q)_n + \sum_{k=1}^{n-1} (x; q)_k (a/x; q)_k (1 - q^n) \sum_{j=1}^{n-k} (-1)^j \begin{bmatrix} n-k-1 \\ j-1 \end{bmatrix} \begin{bmatrix} k+j-1 \\ j-1 \end{bmatrix} \frac{q^{\binom{j}{2} + kj} a^j}{1 - q^j}. \quad (4.5)$$

Proof. We first prove (4.4). Taking $x = q^{-m}$ ($0 \leq m \leq n-1$), we have

$$\begin{aligned} & \sum_{k=0}^{n-1} \frac{(x; q)_k (a/x; q)_k (1 - q^n)}{(q; q)_k (1 - q^{n-k})} \sum_{j=0}^k (-1)^j \begin{bmatrix} k \\ j \end{bmatrix} q^{\binom{j}{2}} (aq^{k+j}; q)_{n-k} \\ &= \sum_{k=0}^{n-1} \frac{(q^{-m}; q)_k (aq^m; q)_k (1 - q^n)}{(q; q)_k (1 - q^{n-k})} \sum_{j=0}^k (-1)^j \begin{bmatrix} k \\ j \end{bmatrix} q^{\binom{j}{2}} (aq^{k+j}; q)_{n-k} \\ &= \sum_{k=0}^m (-1)^k \begin{bmatrix} m \\ k \end{bmatrix} q^{\binom{k}{2} - mk} \frac{(aq^m; q)_k (1 - q^n)}{(1 - q^{n-k})} \sum_{j=0}^k (-1)^j \begin{bmatrix} k \\ j \end{bmatrix} q^{\binom{j}{2}} \frac{(aq^j; q)_n}{(aq^j; q)_k} \\ &= \sum_{j=0}^m (-1)^j \begin{bmatrix} m \\ j \end{bmatrix} q^{\binom{j}{2}} (aq^j; q)_n \sum_{k=j}^m (-1)^k \begin{bmatrix} m-j \\ k-j \end{bmatrix} q^{\binom{k}{2} - mk} \frac{(aq^m; q)_k (1 - q^n)}{(aq^j; q)_k (1 - q^{n-k})}. \end{aligned} \quad (4.6)$$

It follows from (4.1) that

$$\begin{aligned} & \sum_{k=j}^m (-1)^k \begin{bmatrix} m-j \\ k-j \end{bmatrix} q^{\binom{k}{2} - mk} \frac{(aq^m; q)_k (1 - q^n)}{(aq^j; q)_k (1 - q^{n-k})} \\ &= \frac{(aq^m; q)_j}{(aq^j; q)_j} \sum_{k=j}^m (-1)^k \begin{bmatrix} m-j \\ k-j \end{bmatrix} q^{\binom{m-k+1}{2} - \binom{m+1}{2}} \frac{(aq^{m+j}; q)_{k-j} (1 - q^n)}{(aq^{2j}; q)_{k-j} (1 - q^{n-k})} \\ &= \frac{(-1)^m (aq^m; q)_j (aq^{n+j}; q)_{m-j} (q; q)_{m-j} (1 - q^n) q^{-\binom{m+1}{2}}}{(aq^j; q)_j (aq^{2j}; q)_{m-j} (q^{n-m}; q)_{m-j+1}} \\ &= \frac{(-1)^m (a; q)_j (aq^{n+j}; q)_{m-j} (q; q)_{m-j} (1 - q^n) q^{-\binom{m+1}{2}}}{(a; q)_m (q^{n-m}; q)_{m-j+1}}. \end{aligned}$$

Therefore, the right-hand side of (4.6) can be simplified as

$$\frac{(a; q)_{m+n}(1-q^n)q^{-\binom{m+1}{2}}}{(a; q)_m} \sum_{j=0}^m (-1)^{m-j} \begin{bmatrix} m \\ j \end{bmatrix} q^{\binom{j}{2}} \frac{(q; q)_{m-j}}{(q^{n-m}; q)_{m-j+1}} = (aq^m; q)_n, \quad (4.7)$$

where the last equality follows from (4.2). Noticing that $(q^{-m}; q)_n = 0$ for $0 \leq m \leq n-1$, we have proved that both sides of (4.4) are equal for $x = q^{-m}$ ($0 \leq m \leq n-1$), and by symmetry, for $x = aq^m$ ($0 \leq m \leq n-1$) too. Furthermore, both sides of (4.4) are of the form $x^{-n}P(x)$ with $P(x)$ being a polynomial in x of degree $2n$ with the leading coefficient $(-1)^n q^{\binom{n}{2}}$. Hence, they must be identical. This proves (4.4).

By the q -binomial theorem (see, for example, [4, Theorem 3.3]), for $k \geq 1$, we have

$$\begin{aligned} \sum_{j=0}^k (-1)^j \begin{bmatrix} k \\ j \end{bmatrix} q^{\binom{j}{2}} (aq^{k+j}; q)_{n-k} &= \sum_{j=0}^k (-1)^j \begin{bmatrix} k \\ j \end{bmatrix} q^{\binom{j}{2}} \sum_{i=0}^{n-k} (-1)^i \begin{bmatrix} n-k \\ i \end{bmatrix} q^{\binom{i}{2}+(k+j)i} a^i \\ &= \sum_{i=0}^{n-k} (-1)^i \begin{bmatrix} n-k \\ i \end{bmatrix} q^{\binom{i}{2}+ik} a^i \sum_{j=0}^k (-1)^j \begin{bmatrix} k \\ j \end{bmatrix} q^{\binom{j}{2}+ij} \\ &= \sum_{i=1}^{n-k} (-1)^i \begin{bmatrix} n-k \\ i \end{bmatrix} (q^i; q)_k q^{\binom{i}{2}+ik} a^i. \end{aligned} \quad (4.8)$$

Moreover, for $k = 0$, the left-hand side of (4.8) is clearly equal to $(a; q)_n$. Noticing that

$$\begin{bmatrix} n-k \\ i \end{bmatrix} \frac{(q^i; q)_k}{(q; q)_k (1-q^{n-k})} = \begin{bmatrix} n-k-1 \\ i-1 \end{bmatrix} \begin{bmatrix} k+i-1 \\ i-1 \end{bmatrix} \frac{1}{1-q^i},$$

we complete the proof of (4.5). \square

Lemma 4.3. *Let n and h be positive integers and let m and s be nonnegative integers with $h \leq n-m$ and $s \leq m$. Then*

$$\begin{aligned} &\sum_{j=s}^m \sum_{k=s}^n \frac{(q^{-n}; q)_j (q^{-n}; q)_k (x; q)_j (x; q)_k (q^{j-m-h+1}; q)_{h-1} (q^{k-m-h+1}; q)_{h-1} (1-q^{k-j}) q^{2j+k}}{(q; q)_{j-s} (q; q)_{j+s} (q; q)_{k-s} (q; q)_{k+s}} \\ &= \frac{(q; q)_n^2 (q; q)_{h-1} (x; q)_s (x; q)_{m+h} (q^{s+1}/x; q)_{n-s-h} x^{n-s-h} q^{\frac{m^2+3m-s^2+s}{2}-mn-mh-h^2+h}}{(-1)^{m-s-1} (q; q)_{m-s} (q; q)_{m+s} (q; q)_{n-s} (q; q)_{n+s} (q; q)_{n-m-h}}. \end{aligned} \quad (4.9)$$

Proof. Note that both sides of (4.9) are polynomial in x of degree $m+n$ with the same leading coefficient. Therefore, to prove (4.9), it suffices to prove that both sides have the same roots as polynomials in x . Denote the left-hand side of (4.9) by $L_{m,n}(x)$. We first assert that

$$\begin{aligned} L_{m,n}(x) &= \sum_{j=s}^m \sum_{k=m+1}^n \frac{(q^{-n}; q)_j (q^{-n}; q)_k (x; q)_j (x; q)_k}{(q; q)_{j-s} (q; q)_{j+s} (q; q)_{k-s} (q; q)_{k+s}} \\ &\quad \times (q^{j-m-h+1}; q)_{h-1} (q^{k-m-h+1}; q)_{h-1} (1-q^{k-j}) q^{2j+k}. \end{aligned} \quad (4.10)$$

In fact, since $(1-q^{j-k})q^{2k+j} = -(1-q^{k-j})q^{k+2j}$, the double sum $\sum_{j=s}^m \sum_{k=s}^m$ for the same summand in (4.10) is equal to 0. We now consider the following three cases.

- If $s \geq 1$, then it is easily seen that $(x; q)_s^2$ divides $L_{m,n}(x)$, which means that the numbers q^{-r} ($0 \leq r \leq s-1$) are roots of $L_{m,n}(x)$ with multiplicity 2.
- For $x = q^{-r}$ with $s \leq r \leq m+h-1$, we have

$$L_{m,n}(q^{-r}) = \sum_{j=s}^m \sum_{k=s}^n \frac{(q^{-n}; q)_j (q^{-n}; q)_k (q^{-r}; q)_j (q^{-r}; q)_k}{(q; q)_{j-s} (q; q)_{j+s} (q; q)_{k-s} (q; q)_{k+s}} \\ \times (q^{j-m-h+1}; q)_{h-1} (q^{k-m-h+1}; q)_{h-1} (1 - q^{k-j}) q^{2j+k}.$$

If $r \leq m$, then $L_{m,n}(q^{-r}) = L_{m,r}(q^{-n}) = 0$ by the antisymmetry of j and k in $L_{m,r}(q^{-n})$. If $r \geq m+1$, then $h \geq r-m+1$, i.e., $r-m-h+1 \leq 0$, and so $(q^{k-m-h+1}; q)_{h-1} = 0$ for $m+1 \leq k \leq r$. Hence, by (4.10), we again get $L_{m,n}(q^{-r}) = L_{m,r}(q^{-n}) = 0$.

- For $x = q^r$ with $s+1 \leq r \leq n-h$, we shall prove that

$$\sum_{k=s}^n \frac{(q^{-n}; q)_k (q^r; q)_k (q^{k-m-h+1}; q)_{h-1} (1 - q^{k-j}) q^{2j+k}}{(q; q)_{k-s} (q; q)_{k+s}} = 0. \quad (4.11)$$

In fact, the left-hand side of (4.11) can be written as

$$(q^{-n}; q)_s (q^r; q)_s \sum_{k=s}^n \frac{(q^{-n+s}; q)_{k-s} (q^{r+s}; q)_{k-s} (q^{k-m-h+1}; q)_{h-1} (1 - q^{k-j}) q^{2j+k}}{(q; q)_{k-s} (q; q)_{k+s}} \\ = (q^{-n}; q)_s (q^r; q)_s \sum_{k=0}^{n-s} (-1)^k \begin{bmatrix} n-s \\ k \end{bmatrix} q^{-(n-s)k + \binom{k}{2}} R_k, \quad (4.12)$$

where

$$R_k = \frac{(q^{r+s}; q)_k (q^{k+s-m-h+1}; q)_{h-1} (1 - q^{k+s-j}) q^{2j+k+s}}{(q; q)_{k+2s}}.$$

Since

$$\frac{(q^{r+s}; q)_k}{(q; q)_{k+2s}} = \frac{(q^{k+2s+1}; q)_{r-s-1}}{(q; q)_{r+s-1}},$$

we see that R_k is a polynomial in q^k of degree $r-s-1+h-1+2 \leq n-s$ with constant term 0. By the finite q -binomial theorem, see [4, Theorem 3.3],

$$\sum_{k=0}^n (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k+1}{2}} x^k = (xq; q)_n, \quad (4.13)$$

we have

$$\sum_{k=0}^n (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k+1}{2} - ik} = \begin{cases} 0, & \text{for } 1 \leq i \leq n, \\ (q; q)_n, & \text{for } i = 0. \end{cases} \quad (4.14)$$

It follows that the right-hand side of (4.12) is equal to 0. Namely, the identity (4.11) holds.

Thus, we have found out all the $m + n$ roots of $L_{m,n}(x)$, which are clearly the same as those of the right-hand side of (4.9). This completes the proof. \square

Remark. The authors [15] utilized the identity (4.14) to give a short proof of Jackson's terminating q -analogue of Dixon's identity [8, 18]:

$$\sum_{k=-a}^a (-1)^k q^{\frac{3k^2+k}{2}} \begin{bmatrix} a+b \\ a+k \end{bmatrix} \begin{bmatrix} b+c \\ b+k \end{bmatrix} \begin{bmatrix} c+a \\ c+k \end{bmatrix} = \begin{bmatrix} a+b+c \\ a+b \end{bmatrix} \begin{bmatrix} a+b \\ a \end{bmatrix}.$$

Lemma 4.4. *Let n and h be positive integers and let m and s be nonnegative integers with $h \leq n - m$ and $s \leq m$. Then*

$$\begin{aligned} & \sum_{j=s}^m \sum_{k=m+h}^n \frac{(q^{-2n}; q^2)_j (q^{-2n}; q^2)_k (1 - q^{k-j}) q^{j+k+jh} \begin{bmatrix} k-m-1 \\ h-1 \end{bmatrix} \begin{bmatrix} m+h-j-1 \\ h-1 \end{bmatrix}}{(q; q)_{j-s} (q; q)_{j+s} (q; q)_{k-s} (q; q)_{k+s}} \\ &= \frac{(-1)^{n-m-h} (q^2; q^2)_n^2 (-q; q)_{2n-h} q^{m^2-n^2-2mn+mh}}{(q; q)_{m-s} (q; q)_{m+s} (q^2; q^2)_{n-s} (q^2; q^2)_{n+s} (q; q)_{h-1} (q^2, q^2)_{n-m-h}}. \end{aligned} \quad (4.15)$$

Proof. By the definition of q -binomial coefficients, there holds $\begin{bmatrix} k-m-1 \\ h-1 \end{bmatrix} = 0$ for $m+1 \leq k < m+h$. Hence, the left-hand side of (4.15) remains unchanged if we replace $\sum_{k=m+h}^n$ by $\sum_{k=m+1}^n$. Furthermore, we have

$$\begin{bmatrix} k-m-1 \\ h-1 \end{bmatrix} \begin{bmatrix} m+h-j-1 \\ h-1 \end{bmatrix} = \frac{(q^{j-m-h+1}; q)_{h-1} (q^{k-m-h+1}; q)_{h-1} q^{(m-j)(h-1) - \binom{h}{2}}}{(-1)^{h-1} (q; q)_{h-1}^2}$$

The proof then follows from the identity (4.9) with $x = -q^{-n}$. \square

Proof of Theorem 2.5. The left-hand side of (2.9) may be expanded as

$$\begin{aligned} & \sum_{k=s}^n \frac{(q^{-2n}; q^2)_k^2 (x; q)_k^2 q^{2k}}{(q; q)_{k-s}^2 (q; q)_{k+s}^2} (x; q)_k (q/x; q)_k \\ &+ \sum_{s \leq j < k \leq n} \frac{(q^{-2n}; q^2)_j (q^{-2n}; q^2)_k q^{j+k} ((x; q)_j (q/x; q)_k + (x; q)_k (q/x; q)_j)}{(q; q)_{j-s} (q; q)_{j+s} (q; q)_{k-s} (q; q)_{k+s}}. \end{aligned} \quad (4.16)$$

For $0 \leq j < k$, from (4.5) we deduce that

$$\begin{aligned} & (x; q)_j (q/x; q)_k + (x; q)_k (q/x; q)_j \\ &= (x; q)_j (q/x; q)_j ((xq^j; q)_{k-j} + (q^{j+1}/x; q)_{k-j}) \\ &= (x; q)_k (q/x; q)_k + (x; q)_j (q/x; q)_j (q^{2j+1}; q)_{k-j} \\ &+ \sum_{i=1}^{k-j-1} (x; q)_{j+i} (q/x; q)_{j+i} (1 - q^{k-j}) \sum_{h=1}^{k-j-i} (-1)^h \begin{bmatrix} k-j-i-1 \\ h-1 \end{bmatrix} \begin{bmatrix} i+h-1 \\ h-1 \end{bmatrix} \frac{q^{\binom{h+1}{2} + (i+2j)h}}{1 - q^h}, \\ &= (x; q)_k (q/x; q)_k + (x; q)_j (q/x; q)_j \\ &+ \sum_{i=0}^{k-j-1} (x; q)_{j+i} (q/x; q)_{j+i} (1 - q^{k-j}) \sum_{h=1}^{k-j-i} (-1)^h \begin{bmatrix} k-j-i-1 \\ h-1 \end{bmatrix} \begin{bmatrix} i+h-1 \\ h-1 \end{bmatrix} \frac{q^{\binom{h+1}{2} + (i+2j)h}}{1 - q^h}, \end{aligned} \quad (4.17)$$

where in the last step we have used the q -binomial theorem:

$$(q^{2j+1}; q)_{k-j} = 1 + \sum_{h=1}^{k-j} (-1)^h \begin{bmatrix} k-j \\ h \end{bmatrix} q^{\binom{h+1}{2} + 2jh}.$$

By (4.17), we may write (4.16) as $\sum_{m=s}^n a_m(x; q)_m(q/x; q)_m$, where

$$\begin{aligned} a_m &= \sum_{j=s}^n \frac{(q^{-2n}; q^2)_j (q^{-2n}; q^2)_m q^{j+m}}{(q; q)_{j-s} (q; q)_{j+s} (q; q)_{m-s} (q; q)_{m+s}} \\ &\quad + \sum_{j=s}^m \sum_{k=m+1}^n \frac{(q^{-2n}; q^2)_j (q^{-2n}; q^2)_k (1 - q^{k-j}) q^{j+k}}{(q; q)_{j-s} (q; q)_{j+s} (q; q)_{k-s} (q; q)_{k+s}} \\ &\quad \times \sum_{h=1}^{k-m} (-1)^h \begin{bmatrix} k-m-1 \\ h-1 \end{bmatrix} \begin{bmatrix} m+h-j-1 \\ h-1 \end{bmatrix} \frac{q^{\binom{h+1}{2} + (m+j)h}}{1 - q^h}. \end{aligned} \quad (4.18)$$

It is easy to see that

$$\begin{aligned} \sum_{j=s}^n \frac{(q^{-2n}; q^2)_j q^j}{(q; q)_{j-s} (q; q)_{j+s}} &= \frac{(q^{-2n}; q^2)_s q^s}{(q; q)_{2s}} \sum_{j=s}^n \frac{(q^{-2n+2s}; q^2)_j q^{j-s}}{(q; q)_{j-s} (q^{2s+1}; q)_{j-s}} \\ &= \frac{(q^{-2n}; q^2)_s q^s}{(q; q)_{2s}} {}_2\phi_1 \left[\begin{matrix} q^{-n+s}, -q^{-n+s} \\ q^{2s+1} \end{matrix}; q, q \right] \\ &= (-1)^{n-s} \frac{(q^{-2n}; q^2)_s (-q^{n+s+1}; q)_{n-s} q^{s-(n-s)^2}}{(q; q)_{2s} (q^{2s+1}; q)_{n-s}} \end{aligned}$$

by the q -Chu-Vandermonde summation formula [13, Appendix (II.6)]. Hence,

$$\begin{aligned} &\sum_{j=s}^n \frac{(q^{-2n}; q^2)_j (q^{-2n}; q^2)_m q^{j+m}}{(q; q)_{j-s} (q; q)_{j+s} (q; q)_{m-s} (q; q)_{m+s}} \\ &= (-1)^{n-s} \frac{(q^{-2n}; q^2)_s (q^{-2n}; q^2)_m (-q^{n+s+1}; q)_{n-s} q^{m+s-(n-s)^2}}{(q; q)_{m-s} (q; q)_{m+s} (q; q)_{n+s}} \\ &= \frac{(-1)^{n-m} (q^2; q^2)_n^2 (-q; q)_{2n} q^{m^2-n^2-2mn}}{(q; q)_{m-s} (q; q)_{m+s} (q^2; q^2)_{n-s} (q^2; q^2)_{n+s} (q^2, q^2)_{n-m}}. \end{aligned} \quad (4.19)$$

Substituting (4.19) and (4.15) into (4.18), we obtain

$$a_m = \frac{(-1)^{n-m} (q^2; q^2)_n^2 q^{m^2-n^2-2mn}}{(q; q)_{m-s} (q; q)_{m+s} (q^2; q^2)_{n-s} (q^2; q^2)_{n+s}} \sum_{h=0}^{n-m} \frac{(-1)^h (-q; q)_{2n-h} q^{\binom{h+1}{2} + 2mh}}{(q; q)_h (q^2, q^2)_{n-m-h}}. \quad (4.20)$$

Replacing h by $n - m - h$, we have

$$\begin{aligned}
& \sum_{h=0}^{n-m} \frac{(-1)^h (-q; q)_{2n-h} q^{\binom{h+1}{2} + 2mh}}{(q; q)_h (q^2, q^2)_{n-m-h}} \\
&= \frac{(-q; q)_{n+m}}{(q; q)_{n-m}} (-1)^{n-m} q^{\binom{n-m+1}{2} + 2m(n-m)} \sum_{h=0}^{n-m} \frac{(q^{m-n}; q)_h (-q^{n+m+1}; q)_h}{(-q; q)_h (q; q)_h} q^{-2hm} \\
&= \frac{(-q; q)_{n+m}}{(q; q)_{n-m}} (-1)^{n-m} q^{\binom{n-m+1}{2} + 2m(n-m)} {}_2\phi_1 \left[\begin{matrix} q^{-(n-m)}, -q^{m+n+1} \\ -q \end{matrix}; q, q^{-2m} \right] \\
&= \frac{(-q; q)_{n+m} (q^{-n-m}; q)_{n-m}}{(q; q)_{n-m} (-q; q)_{n-m}} (-1)^{n-m} q^{\binom{n-m+1}{2} + 2m(n-m)} \\
&= \frac{(q^2; q^2)_{m+n}}{(q^2; q^2)_{n-m} (q; q)_{2m}}, \tag{4.21}
\end{aligned}$$

where we have used the q -Chu-Vandermonde summation formula [13, Appendix (II.7)]. It follows from (4.20) and (4.21) that a_m is just the coefficient of $(x; q)_m (q/x; q)_m$ in the right-hand side of (2.9). This completes the proof. \square

5. Proof of Theorem 2.3

We first give a congruence modulo $[p]$.

Lemma 5.1. *Let p be an odd prime and m, r two positive integers with $p \nmid m$. Let $s \leq \min\{\langle -\frac{r}{m} \rangle_p, \langle -\frac{m-r}{m} \rangle_p\}$ be a nonnegative integer. Then the following congruence holds modulo $[p]$:*

$$\begin{aligned}
& \sum_{k=s}^{\frac{p-1}{2}} \frac{(q^m; q^{2m})_k (q^r; q^m)_k q^{mk}}{(q^m; q^m)_{k-s} (q^m; q^m)_{k+s}} \\
&\equiv \begin{cases} \frac{q^{\frac{(\langle -\frac{r}{m} \rangle_p + s)m}{2}} (q^m; q^{2m})_{\frac{\langle -\frac{r}{m} \rangle_p - s}{2}} (q^{-m\langle -\frac{r}{m} \rangle_p}; q^m)_s}{(q^{2m}; q^{2m})_{\frac{\langle -\frac{r}{m} \rangle_p + s}{2}}}, & \text{if } \langle -\frac{r}{m} \rangle_p \equiv s \pmod{2}, \\ 0 & \text{if } \langle -\frac{r}{m} \rangle_p \equiv s+1 \pmod{2}. \end{cases} \tag{5.1}
\end{aligned}$$

Proof. It is easy to see that $\langle -\frac{r}{m} \rangle_p + \langle -\frac{m-r}{m} \rangle_p = p-1$, and so $s \leq \frac{p-1}{2}$. Since p is an odd prime, we see that $(q^m; q^{2m})_k \equiv 0 \pmod{[p]}$ for $\frac{p+1}{2} \leq k \leq p-s-1$, which means that

$$\sum_{k=s}^{\frac{p-1}{2}} \frac{(q^m; q^{2m})_k (q^r; q^m)_k q^{mk}}{(q^m; q^m)_{k-s} (q^m; q^m)_{k+s}} \equiv \sum_{k=s}^{p-s-1} \frac{(q^m; q^{2m})_k (q^r; q^m)_k q^{mk}}{(q^m; q^m)_{k-s} (q^m; q^m)_{k+s}} \pmod{[p]}.$$

Let $a = \frac{m\langle -\frac{r}{m} \rangle_p + r}{p}$. Then $m \mid r - ap$ and $r - ap = -m\langle -\frac{r}{m} \rangle_p \leq 0$. It is clear that $(q^r; q^m)_k \equiv (q^{r-ap}; q^m)_k \pmod{[p]}$ and $(q^{r-ap}; q^m)_k = 0$ for $k > \langle -\frac{r}{m} \rangle_p$. Moreover, we

have $p - s - 1 \geq p - \langle -\frac{m-r}{m} \rangle_p - 1 = \langle -\frac{r}{m} \rangle_p \geq s$, and therefore,

$$\begin{aligned}
\sum_{k=s}^{p-s-1} \frac{(q^m; q^{2m})_k (q^r; q^m)_k q^{mk}}{(q^m; q^m)_{k-s} (q; q)_{k+s}} &\equiv \sum_{k=s}^{p-s-1} \frac{(q^m; q^{2m})_k (q^{r-ap}; q^m)_k q^{mk}}{(q^m; q^m)_{k-s} (q; q)_{k+s}} \\
&= \sum_{k=s}^{\langle -\frac{r}{m} \rangle_p} \frac{(q^m; q^{2m})_k (q^{r-ap}; q^m)_k q^{mk}}{(q^m; q^m)_{k-s} (q^m; q^m)_{k+s}} \\
&= \frac{(q^m; q^{2m})_s (q^{-m\langle -\frac{r}{m} \rangle_p}; q^m)_s q^{ms}}{(q^m; q^m)_{2s}} \\
&\quad \times {}_3\phi_2 \left[\begin{matrix} q^{-m(\langle -\frac{r}{m} \rangle_p - s)}, q^{(s+\frac{1}{2})m}, -q^{(s+\frac{1}{2})m} \\ 0, q^{(2s+1)m} \end{matrix}; q^m, q^m \right] \pmod{[p]}.
\end{aligned}$$

The proof then follows from Andrew's identity (3.3). \square

Proof of Theorems 2.3. By Lemma 3.1, for $0 \leq k \leq \frac{p-1}{2}$, we have

$$\begin{aligned}
\frac{(q^m; q^m)_{2k}}{(q^{2m}; q^{2m})_k^2} &= \left[\begin{matrix} 2k \\ k \end{matrix} \right]_{q^{2m}} \frac{1}{(-q^m; q^m)_{2k}} \\
&\equiv (-1)^k q^{mk^2 - mkp} \left[\begin{matrix} \frac{p-1}{2} + k \\ 2k \end{matrix} \right]_{q^{2m}} (-q^m; q^m)_{2k} \\
&= \frac{(-1)^k q^{mk^2 - mkp} (q^{2m}; q^{2m})_{\frac{p-1}{2} + k}}{(q^{2m}; q^{2m})_{\frac{p-1}{2} - k} (q^m; q^m)_{2k}} \pmod{[p]^2},
\end{aligned}$$

and so

$$\begin{aligned}
&\sum_{k=s}^{\frac{p-1}{2}} \frac{(q^m; q^m)_{2k} (q^r; q^m)_k (q^{m-r}; q^m)_k q^{mk}}{(q^m; q^m)_{k-s} (q^m; q^m)_{k+s} (q^{2m}; q^{2m})_k^2} \\
&\equiv \sum_{k=s}^{\frac{p-1}{2}} \frac{(-1)^k (q^m; q^m)_{\frac{p-1}{2} + k} (q^r; q^m)_k (q^{m-r}; q^m)_k q^{mk^2 - mk(p-1)}}{(q^{2m}; q^{2m})_{\frac{p-1}{2} - k} (q^m; q^m)_{k-s} (q^m; q^m)_{k+s} (q^m; q^m)_{2k}} \pmod{[p]^2}. \tag{5.2}
\end{aligned}$$

Letting $q \rightarrow q^m$, $x = q^r$ and $n = \frac{p-1}{2}$ in Theorem 2.5, we see that the right-hand side of (5.2) can be written as

$$\begin{aligned}
&\frac{(-1)^{\frac{p-1}{2}} (q^{2m}; q^{2m})_{\frac{p-1}{2} - s} (q^{2m}; q^{2m})_{\frac{p-1}{2} + s} q^{\frac{(p-1)^2}{4}}}{(q^{2m}; q^{2m})_{\frac{p-1}{2}}^2} \left(\sum_{k=s}^{\frac{p-1}{2}} \frac{(q^{m(1-p)}; q^{2m})_k (q^r; q^m)_k q^{mk}}{(q^m; q^m)_{k-s} (q^m; q^m)_{k+s}} \right) \\
&\quad \times \left(\sum_{k=s}^{\frac{p-1}{2}} \frac{(q^{m(1-p)}; q^{2m})_k (q^{m-r}; q^m)_k q^{mk}}{(q^m; q^m)_{k-s} (q^m; q^m)_{k+s}} \right). \tag{5.3}
\end{aligned}$$

If $\langle -\frac{r}{m} \rangle_p \equiv s + 1 \pmod{2}$, then by the congruence (5.1), we have

$$\sum_{k=s}^{\frac{p-1}{2}} \frac{(q^{m(1-p)}; q^{2m})_k (q^r; q^m)_k q^{mk}}{(q^m; q^m)_{k-s} (q^m; q^m)_{k+s}} \equiv \sum_{k=s}^{\frac{p-1}{2}} \frac{(q^m; q^{2m})_k (q^r; q^m)_k q^{mk}}{(q^m; q^m)_{k-s} (q^m; q^m)_{k+s}} \equiv 0 \pmod{[p]},$$

and also $\langle -\frac{m-r}{m} \rangle_p \equiv s + 1 \pmod{2}$ which means that

$$\sum_{k=s}^{\frac{p-1}{2}} \frac{(q^{m(1-p)}; q^{2m})_k (q^{m-r}; q^m)_k q^{mk}}{(q^m; q^m)_{k-s} (q^m; q^m)_{k+s}} \equiv 0 \pmod{[p]}.$$

Noticing the fact $(q^{2m}; q^{2m})_{\frac{p-1}{2}} \not\equiv 0 \pmod{[p]}$, we conclude that the right-hand side of (5.2) is congruent to 0 modulo $[p]^2$. This proves (2.5).

To prove (2.6), just observe that (see the proof of Lemma 5.1)

$$(q^r; q^m)_k \equiv (q^{m-r}; q^m)_k \equiv 0 \pmod{[p]}$$

for $\max \{ \langle -\frac{r}{m} \rangle_p, \langle -\frac{m-r}{m} \rangle_p \} < k \leq p-1$, and

$$(q^r; q^m)_k (q^{m-r}; q^m)_k \equiv \frac{(q^m; q^m)_{2k}}{(q^m; q^m)_{k-s} (q^m; q^m)_{k+s}} \equiv 0 \pmod{[p]}.$$

for $\frac{p-1}{2} < k \leq \max \{ \langle -\frac{r}{m} \rangle_p, \langle -\frac{m-r}{m} \rangle_p \}$.

Finally, the proof of (2.7) follows from factorizing (5.2) into (5.3), applying the first case of the congruence (5.1), and then using the aforementioned relation $\langle -\frac{r}{m} \rangle_p + \langle -\frac{m-r}{m} \rangle_p = p-1$. \square

6. Proof of Theorems 2.6 and 2.7

The following lemma can be derived from the q -Chu-Vandermonde formula if the sums are written in standard basic hypergeometric series. Here we give a different proof.

Lemma 6.1. *Let n and s be nonnegative integers with $s \leq n$. Then*

$$\sum_{k=0}^n (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} m+k \\ n \end{bmatrix} q^{\binom{k}{2}-nk} = (-1)^n q^{-\binom{n+1}{2}}, \quad (6.1)$$

$$\sum_{k=0}^n (-1)^k \begin{bmatrix} n+k \\ 2k \end{bmatrix}_{q^2} \begin{bmatrix} 2k+2s \\ k+s \end{bmatrix}_{q^2} \frac{q^{k^2-k-2nk}}{(-q^{2k+1}; q)_{2s}} = (-1)^n q^{-n(n+1)}. \quad (6.2)$$

Proof. It is not difficult to see that the two identities (6.1) and (6.2) are equivalent, respectively, to

$$\sum_{k=0}^n (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} m+n-k \\ n \end{bmatrix} q^{\binom{k+1}{2}} = 1, \quad (6.3)$$

$$\sum_{k=0}^n (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_{q^2} \begin{bmatrix} 2n-k \\ n \end{bmatrix}_{q^2} \frac{(q^{2n-2k+1}; q^2)_s}{(q^{2n-2k+2}; q^2)_s} q^{\binom{k+1}{2}} = 1. \quad (6.4)$$

Since $\begin{bmatrix} m+n-k \\ n \end{bmatrix}$ can be written as a polynomial in q^{-k} of degree n with constant term $1/(q; q)_n$. Identity (6.3) then follows from (4.14). On the other hand, since $0 \leq s \leq n$, we see that

$$\begin{bmatrix} 2n-k \\ n \end{bmatrix}_{q^2} \frac{(q^{2n-2k+1}; q^2)_s}{(q^{2n-2k+2}; q^2)_s} = \frac{(q^{2n-2k+2s+2}; q^2)_{n-s} (q^{2n-2k+1}; q^2)_s}{(q^2; q^2)_n}$$

is a polynomial in q^{-2k} of degree n with constant term $\frac{1}{(q^2; q^2)_n}$. Therefore, identity (6.4) follows from (4.14) with $q \rightarrow q^2$. This completes the proof. \square

Proof of Theorem 2.6. It is easy to see that

$$\frac{(q; q^2)_k}{(q^2; q^2)_k} = \begin{bmatrix} 2k \\ k \end{bmatrix}_{q^2} \frac{1}{(-q; q)_{2k}}.$$

Hence, by Lemmas 3.1 and 6.1, we have

$$\begin{aligned} \sum_{k=0}^{\frac{p-1}{2}} \frac{(q; q^2)_k (q; q^2)_{k+s}}{(q^2; q^2)_k (q^2; q^2)_{k+s}} &= \sum_{k=0}^{\frac{p-1}{2}} \begin{bmatrix} 2k \\ k \end{bmatrix}_{q^2} \begin{bmatrix} 2k+2s \\ k+s \end{bmatrix}_{q^2} \frac{1}{(-q; q)_{2k} (-q; q)_{2k+2s}} \\ &\equiv \sum_{k=0}^{\frac{p-1}{2}} (-1)^k \begin{bmatrix} \frac{p-1}{2} + k \\ 2k \end{bmatrix}_{q^2} \begin{bmatrix} 2k+2s \\ k+s \end{bmatrix}_{q^2} \frac{(-q; q)_{2k} q^{k^2-kp}}{(-q; q)_{2k+2s}} \\ &= (-1)^{\frac{p-1}{2}} q^{\frac{1-p^2}{4}} \pmod{[p]^2}, \end{aligned}$$

as desired. \square

Proof of Theorem 2.7. Again, let $a = \frac{m\langle -\frac{r}{m} \rangle_p + r}{p}$. Then a is a positive integer, $m \mid ps - r$, and so

$$\begin{aligned} \frac{(q^r; q^m)_k}{(q^m; q^m)_k} &= \prod_{j=1}^{k+s} \frac{1 - q^{mj+r-m}}{1 - q^{mj}} \\ &\equiv (-1)^k \prod_{j=1}^k \frac{(1 - q^{ap-mj-r+m}) q^{mj+r-m}}{1 - q^{mj}} \\ &= (-1)^k \begin{bmatrix} \frac{ap-r}{m} \\ k \end{bmatrix}_{q^m} q^{\frac{mk(k-1)}{2} + kr} \\ &\equiv (-1)^k \begin{bmatrix} \langle -\frac{r}{m} \rangle_p \\ k \end{bmatrix}_{q^m} q^{\frac{mk(k-1)}{2} - mk\langle -\frac{r}{m} \rangle_p} \pmod{[p]}, \end{aligned} \tag{6.5}$$

$$\frac{(q^{m-r}; q^m)_{k+s}}{(q^m; q^m)_{k+s}} \equiv \prod_{j=1}^{k+s} \frac{1 - q^{ap+mj-r}}{1 - q^{mj}} = \begin{bmatrix} \frac{ap-r}{m} + k + s \\ k + s \end{bmatrix}_{q^m} = \begin{bmatrix} \langle -\frac{r}{m} \rangle_p + k + s \\ k + s \end{bmatrix}_{q^m} \pmod{[p]}. \tag{6.6}$$

By the congruences (6.5) and (6.6), we have

$$\begin{aligned}
& \sum_{k=0}^{p-s-1} \frac{(q^r; q^m)_k (q^{m-r}; q^m)_{k+s}}{(q^m; q^m)_k (q^m; q^m)_{k+s}} \\
& \equiv \sum_{k=0}^{p-s-1} (-1)^k \begin{bmatrix} \langle -\frac{r}{m} \rangle_p \\ k \end{bmatrix}_{q^m} \begin{bmatrix} \langle -\frac{r}{m} \rangle_p + k + s \\ k + s \end{bmatrix}_{q^m} q^{\frac{mk(k-1)}{2} - mk \langle -\frac{r}{m} \rangle_p} \\
& = (-1)^{\langle -\frac{r}{m} \rangle_p} q^{\frac{-m \langle -\frac{r}{m} \rangle_p (\langle -\frac{r}{m} \rangle_p + 1)}{2}} \pmod{[p]},
\end{aligned}$$

where in the last step we have used $p-s-1 \geq p - \langle -\frac{m-r}{m} \rangle_p - 1 = \langle -\frac{r}{m} \rangle_p$ and the identity (6.1). This proves (2.11).

To prove (2.12), just notice that if $p \equiv \pm 1 \pmod{m}$, then $\frac{r(m-r)(1-p^2)}{2m}$ is an integer and

$$\frac{-m \langle -\frac{r}{m} \rangle_p (\langle -\frac{r}{m} \rangle_p + 1)}{2} \equiv \frac{r(m-r)(1-p^2)}{2m} \pmod{p}. \quad \square$$

7. Concluding remarks and open problems

It is natural to ask the following problem:

Problem 7.1. *Are there any q -analogues of Beukers' supercongruence (1.1)?*

Numerical experiments suggest the following companion of Theorem 2.5.

Conjecture 7.2. *Let n and r be nonnegative integers with $r \leq n$. Then*

$$\begin{aligned}
& \left(\sum_{k=r}^n \frac{(q^{-n}; q)_k (x; q^2)_k q^k}{(q; q)_{k-r} (q; q)_{k+r}} \right) \left(\sum_{k=r}^n \frac{(q^{-n}; q)_k (x; q^2)_k q^{(n+1)k - \binom{k}{2}}}{(q; q)_{k-r} (q; q)_{k+r} x^k} \right) \\
& = \frac{(-1)^r (q; q)_n^2 (x; q^2)_r q^r}{(q; q)_{n-r} (q; q)_{n+r} (q^2/x; q^2)_r x^r} \sum_{k=r}^n \frac{(q^{-n}; q)_k (q^{n+1}; q)_k (x; q^2)_k (q^2/x; q^2)_k q^k}{(q; q)_{k-r} (q; q)_{k+r} (q; q)_{2k}}.
\end{aligned}$$

It seems that the congruence (2.6) can be further generalized as follows.

Conjecture 7.3. *Let p be an odd prime and m, r two positive integers with $p \nmid m$. Let $s \leq p-1$ be a nonnegative integer. If $\langle -\frac{r}{m} \rangle_p \equiv s+1 \pmod{2}$, then*

$$\sum_{k=s}^{p-1} \frac{(q^m; q^m)_{2k} (q^r; q^m)_k (q^{m-r}; q^m)_k q^{mk}}{(q^m; q^m)_{k-s} (q^m; q^m)_{k+s} (q^{2m}; q^{2m})_k^2} \equiv 0 \pmod{[p]^2}. \quad (7.1)$$

Note that, if $s > \min\{\langle -\frac{r}{m} \rangle_p, \langle -\frac{m-r}{m} \rangle_p\}$, then the congruence (7.1) is obviously true, since in this case each summand in the left-hand side is congruent to 0 modulo $[p]^2$.

Taking $r = 1$ and $m = 3, 4, 6$ in (7.1), we get

Conjecture 7.4. Let $p \geq 5$ be a prime and let $s \leq p-1$ be a nonnegative integer. Then

$$\begin{aligned} \sum_{k=s}^{p-1} \begin{bmatrix} 2k \\ k+s \end{bmatrix}_{q^3} \frac{(q; q^3)_k (q^2; q^3)_k q^{3k}}{(q^6; q^6)_k^2} &\equiv 0 \pmod{[p]^2}, \quad \text{if } s \equiv \frac{1 + (\frac{-3}{p})}{2} \pmod{2}, \\ \sum_{k=s}^{p-1} \begin{bmatrix} 2k \\ k+s \end{bmatrix}_{q^4} \frac{(q; q^4)_k (q^3; q^4)_k q^{4k}}{(q^8; q^8)_k^2} &\equiv 0 \pmod{[p]^2}, \quad \text{if } s \equiv \frac{1 + (\frac{-2}{p})}{2} \pmod{2}, \\ \sum_{k=s}^{p-1} \begin{bmatrix} 2k \\ k+s \end{bmatrix}_{q^6} \frac{(q; q^6)_k (q^5; q^6)_k q^{6k}}{(q^{12}; q^{12})_k^2} &\equiv 0 \pmod{[p]^2}, \quad \text{if } s \equiv \frac{1 + (\frac{-1}{p})}{2} \pmod{2}. \end{aligned}$$

It is clear that when $q \rightarrow 1$, Conjecture 7.4 reduces to Z.-W. Sun's generalization of (1.4)–(1.6) (see [32, Theorem 1.3(i)]).

We conjecture that Theorem 2.7 and Corollary 2.8 can be further strengthened.

Conjecture 7.5. The congruences (2.13)–(2.15) also hold modulo $[p]^2$.

Conjecture 7.6. Let p be an odd prime and let m, r be positive integers with $p \equiv \pm 1 \pmod{m}$ and $r < m$. Then for any integer s with $0 \leq s \leq \langle -\frac{m-r}{m} \rangle_p$, there holds

$$\sum_{k=0}^{p-s-1} \frac{(q^r; q^m)_k (q^{m-r}; q^m)_{k+s}}{(q^m; q^m)_k (q^m; q^m)_{k+s}} \equiv (-1)^{\langle -\frac{r}{m} \rangle_p} q^{\frac{r(m-r)(1-p^2)}{2m}} \pmod{[p]^2}.$$

Like [16, Conjecture 7.1], Conjecture 7.6 seems have a further generalization as follows:

Conjecture 7.7. Let p be an odd prime and let $m, |r|$ be positive integers with $p \nmid m$ and $m \nmid r$. Then there exists a unique integer $f_{p,m,r}$ such that, for any $0 \leq s \leq \langle -\frac{m-r}{m} \rangle_p$, there holds

$$\sum_{k=0}^{p-s-1} \frac{(q^r; q^m)_k (q^{m-r}; q^m)_{k+s}}{(q^m; q^m)_k (q^m; q^m)_{k+s}} \equiv (-1)^{\langle -\frac{r}{m} \rangle_p} q^{f_{p,m,r}} \pmod{[p]^2}.$$

Furthermore, the numbers $f_{p,m,r}$ satisfy the symmetry $f_{p,m,r} = f_{p,m,m-r}$ and the recurrence relation:

$$f_{p,m,m+r} = \begin{cases} -f_{p,m,r}, & \text{if } r \equiv 0 \pmod{p}, \\ f_{p,m,r} - r, & \text{otherwise.} \end{cases}$$

Here we give some values of $f_{p,m,r}$:

$$\begin{aligned} f_{3,2,1} &= -2, \quad f_{3,2,3} = -3, \quad f_{3,2,5} = 3, \quad f_{3,2,7} = -2, \quad f_{3,2,9} = -9, \quad f_{3,2,11} = 9, \quad f_{3,2,13} = -2, \\ f_{5,3,1} &= -8, \quad f_{5,3,2} = -8, \quad f_{5,3,4} = -9, \quad f_{5,3,5} = -10, \quad f_{5,3,7} = -13, \quad f_{5,3,8} = 10, \quad f_{5,3,10} = -20, \\ f_{5,3,11} &= 2, \quad f_{5,3,13} = 20, \quad f_{5,3,14} = -9, \quad f_{5,3,16} = 7, \quad f_{5,3,17} = -23, \quad f_{5,3,19} = -9, \quad f_{5,8,1} = -23, \\ f_{7,9,1} &= -54, \quad f_{7,9,2} = -21, \quad f_{7,9,4} = -37, \quad f_{7,9,5} = -37, \quad f_{7,9,7} = -21, \quad f_{7,9,8} = -54, \\ f_{7,9,10} &= -55, \quad f_{7,9,11} = -23, \quad f_{7,9,13} = -41, \quad f_{7,9,14} = -42, \quad f_{7,9,16} = -22, \quad f_{7,9,17} = -33. \end{aligned}$$

Acknowledgments. The first author was partially supported by the Fundamental Research Funds for the Central Universities and the National Natural Science Foundation of China (grant 11371144).

References

- [1] S. Ahlgren, Gaussian hypergeometric series and combinatorial congruences, in: Symbolic Computation, Number Theory, Special Functions, Physics and Combinatorics (Gainesville, FI, 1999), Dev. Math., Vol. 4, Kluwer, Dordrecht, 2001, pp. 1–12.
- [2] S. Ahlgren and K. Ono, A Gaussian hypergeometric series evaluation and Apéry number congruences, J. Reine Angew. Math. 518 (2000), 187–212.
- [3] M. Aigner, A Course in Enumeration, Graduate Texts in Mathematics, Vol. 238, Springer, 2007.
- [4] G.E. Andrews, The Theory of Partitions, Cambridge University Press, Cambridge, 1998.
- [5] G.E. Andrews, q -Analogues of the binomial coefficient congruences of Babbage, Wolstenholme and Glaisher, Discrete Math. 204 (1999), 15–25.
- [6] G.E. Andrews, On the q -analogue of Kummer’s theorem and applications, Duke Math. J. 40 (1973) 525–528.
- [7] G.E. Andrews, Applications of basic hypergeometric functions, SIAM Rev. 16 (1974) 441–484.
- [8] W. N. Bailey, A note on certain q -identities, Quart. J. Math., Oxford Ser. 12 (1941), 173–175.
- [9] F. Beukers, Another congruence for the Apéry numbers, J. Number Theory 25 (1987), 201–210.
- [10] P. Candelas, X. de la Ossa, and F. Rodriguez-Villegas, Calabi-Yau manifolds over finite fields I, preprint, 2000, arXiv:hep-th/0012233.
- [11] S. Chowla, B. Dwork, and R. Evans, On the mod p^2 determination of $\binom{(p-1)/2}{(p-1)/4}$, J. Number Theory, 24 (1986), 188–196.
- [12] A. M. Fu and A. Lascoux, q -Identities from Lagrange and Newton interpolation, Adv. Appl. Math. 31 (2003), 527–531.
- [13] G. Gasper and M. Rahman, Basic Hypergeometric Series, Second Edition, Encyclopedia of Mathematics and Its Applications, Vol. 96, Cambridge University Press, Cambridge, 2004.
- [14] J. Greene, Hypergeometric functions over finite fields, Trans. Amer. Math. Soc. 301 (1987), 77–101.
- [15] V.J.W. Guo and J. Zeng, A short proof of the q -Dixon identity, Discrete Math. 296 (2005), 259–261.
- [16] V.J.W. Guo and J. Zeng, Some q -analogues of supercongruences of Rodriguez-Villegas, J. Number Theory, <http://dx.doi.org/10.1016/j.jnt.2014.06.002>
- [17] T. Ishikawa, On Beukers’ congruence, Kobe J. Math. 6 (1989), 49–52.
- [18] F. H. Jackson, Certain q -identities, Quart. J. Math., Oxford Ser. 12 (1941), 167–172.
- [19] D. McCarthy and R. Osburn, A p -adic analogue of a formula of Ramanujan, Arch. Math. 91 (2008), 492–504.
- [20] E. Mortenson, A supercongruence conjecture of Rodriguez-Villegas for a certain truncated hypergeometric function, J. Number Theory 99 (2003), 139–147.
- [21] E. Mortenson, Supercongruences between truncated ${}_2F_1$ hypergeometric functions and their Gaussian analogs, Trans. Amer. Math. Soc. 355 (2003), 987–1007.

- [22] E. Mortenson, Supercongruences for truncated ${}_{n+1}F_n$ hypergeometric series with applications to certain weight three newforms, *Proc. Amer. Math. Soc.* 133 (2005), 321–330.
- [23] H. Pan, A q -analogue of Lehmer’s congruence, *Acta Arith.* 128 (2007), 303–318.
- [24] H. Pan, An elementary approach to $\binom{(p-1)/2}{(p-1)/4}$ modulo p^2 , *Taiwanese J. Math.* 16 (2012), 2197–2202.
- [25] L.-L. Shi and H. Pan, A q -analogue of Wolstenholme’s harmonic series congruence, *Amer. Math. Monthly* 114 (2007), 529–531.
- [26] F. Rodriguez-Villegas, Hypergeometric families of Calabi-Yau manifolds, in: *Calabi-Yau Varieties and Mirror Symmetry* (Toronto, ON, 2001), *Fields Inst. Commun.*, 38, Amer. Math. Soc., Providence, RI, 2003, pp. 223–231.
- [27] A. Straub, A q -analog of Ljunggren’s binomial congruence, in: *23rd International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2011)*, *Discrete Math. Theor. Comput. Sci. Proc.*, AO, Assoc. Discrete Math. Theor. Comput. Sci., Nancy, 2011, pp. 897–902.
- [28] Z.-H. Sun, Congruences concerning Legendre polynomials, *Proc. Amer. Math. Soc.* 139 (2011), 1915–1929.
- [29] Z.-H. Sun, Congruences concerning Legendre polynomials II, *J. Number Theory* 133 (2013), 1950–1976.
- [30] Z.-H. Sun, Generalized Legendre polynomials and related supercongruences, *J. Number Theory* 143 (2014), 293–319.
- [31] Z.-W. Sun, Super congruences and Euler numbers, *Sci. China Math.* 54 (2011), 2509–2535.
- [32] Z.-W. Sun, On sums involving products of three binomial coefficients, *Acta Arith.* 156 (2012) 123–141.
- [33] Z.-W. Sun, Supercongruences involving products of two binomial coefficients, *Finite Fields Appl.* 22 (2013), 24–44.
- [34] R. Tauraso, An elementary proof of a Rodriguez-Villegas supercongruence, preprint, 2009, arXiv:0911.4261v1.
- [35] R. Tauraso, Supercongruences for a truncated hypergeometric series, *Integers* 12 (2012), #A45.
- [36] R. Tauraso, Some q -analogs of congruences for central binomial sums, *Colloq. Math.* 133 (2013), 133–143.
- [37] W. Van Assche, Little q -Legendre polynomials and irrationality of certain Lambert series, *Ramanujan J.* 5 (2001), 295–310.
- [38] L. Van Hamme, Some conjectures concerning partial sums of generalized hypergeometric series, in: *p -Adic Functional Analysis* (Nijmegen, 1996), *Lecture Notes in Pure and Appl. Math.*, Vol. 192, Dekker, 1997, pp. 223–236.
- [39] J. Zeng, On some q -identities related to divisor functions, *Adv. Appl. Math.* 34 (2005), 313–315.